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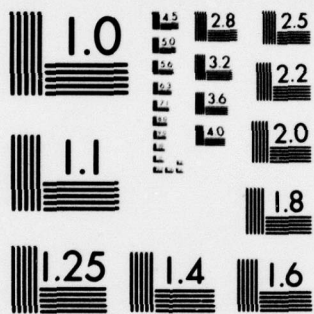
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BICONTINUOUS EXTENSIONS OF INVERTIBLE
COMBINATORIAL FUNCTIONS

Tomaso Toffoli



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Tommaso Toffoli

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AB

BICONTINUOUS EXTENSIONS OF INVERTIBLE COMBINATORIAL FUNCTIONS

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Abstract. We discuss and solve the problem of constructing a diffeomorphic componentwise extension for an arbitrary invertible combinatorial function. Interpreted in physical terms, our solution constitutes a proof of the physical realizability of general computing mechanisms based on *reversible* primitives.

Keywords. Invertible combinatorial functions, continuous extensions, reversibility, Boolean functions.

1. Motivations

In an ordinary digital computer, the two logic states associated with a binary signal are realized as distinguished values of a continuous variable which represents the range of a physical quantity; correspondingly, the logic function associated with a given combinatorial network is realized as the appropriate restriction of a suitable continuous function which characterizes a physical system involving a number of such quantities. If the logic function is not invertible (note that a computation may yield the same output for different inputs), its continuous extension cannot be invertible. On the other hand, the microscopic physical laws which underly the operation of a computer are presumed to be *strictly reversible*, i.e., they uniquely specify a trajectory both forward and backward in time. Thus, it is clear that a noninvertible continuous function such as the above characterizes a physical system only in terms of *statistical* mechanics, rather than of *microscopic* mechanics. In other words, such a function is necessarily an *incomplete* specification of a mechanical system[1]; in particular, it does not give one the means to deal in any detail with the information that is "discarded" during a computation, besides accounting for it in terms of the

increase of a single scalar quantity, the *entropy* of the system[2].

In an attempt to exercise some control on the details of the work-to-heat conversion processes that accompany physical computing (and which are related to the irreversibility of computation), one may conceive of a different approach to the mathematical modeling and the design of computers[3] (see Appendix for a brief summary). In that approach, a major obstacle to arriving at a complete specification of a mechanical system is removed, since computation is modeled exclusively in terms of invertible combinatorial functions. It remains to show that such functions admit in general of a physical realization. This we do in the present paper.

2. Statement of the problem

GOAL 2.1 Given the set $B = \{0,1\}$ and an invertible function $f^{(n)}: B^n \rightarrow B^n$, find a connected manifold $M \supseteq B$ and a diffeomorphism $F^{(n)}: M^n \rightarrow M^n$ such that $f^{(n)}$ is a restriction of $F^{(n)}$.

Our goal can be given the following kinematical interpretation. Consider a box having n input levers and n output levers, as depicted in Figure 2.1 for $n = 2$.

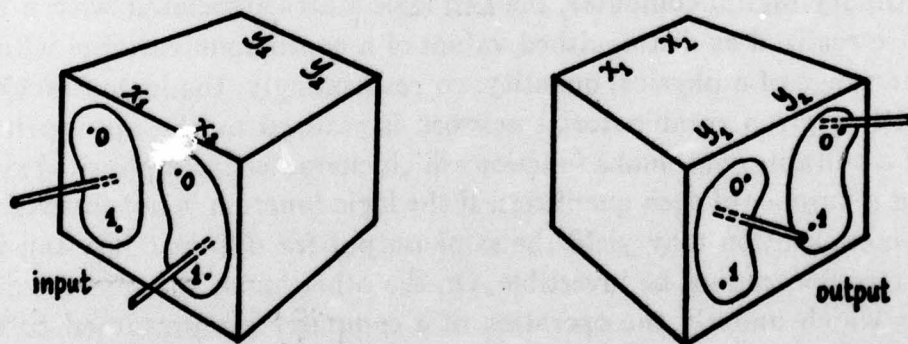


FIG. 2.1 Realization of a combinatorial function by means of continuous mechanisms.

M represents the range of accessible positions for each lever (a manifold is the appropriate mathematical structure for describing this range). Two distinguished positions within M are marked "0" and "1". Assume that the input levers are interconnected to the output ones by means of a passive physical mechanisms (for instance, an assembly of gears, cams, etc.) in such a way that

(a) When all input levers occupy distinguished positions, so do all the output ones. In this way, the box "computes" a combinatorial function from binary n -tuples to binary n -tuples.

(b) The collective configuration of the output levers is a continuous function of the input configuration. Continuity should extend to the higher derivatives (velocity, acceleration, etc.).

(c) The box is *reversible*, i.e., condition (b) holds when input and output levers are exchanged.

Clearly, (c) implies that (a) too holds when input and output levers are exchanged. Thus, the combinatorial function "computed" by the box must be invertible. We want design principles to construct a box with the above properties for any invertible combinatorial function $f^{(n)}$. The specifications for such a box will be represented by a diffeomorphism $F^{(n)}$ from M^n to M^n . (When one is dealing with manifolds instead of intervals of the real line, a *diffeomorphism* is the appropriate generalization of a bicontinuous function).

It must be stressed that Goal 2.1 does not just ask for an arbitrary diffeomorphic extension of the given function $f^{(n)}$ to an arbitrary manifold. Rather, the extension must be *componentwise*. In other words, besides being a *superset* of B^n , the manifold must also be of the form M^n , i.e., possess the same *Cartesian product structure* as B^n ; moreover, the extension itself must maintain the variables *separated*, i.e., each component of the extension must be an extension of the corresponding component of the given function. In physical terms, each binary variable must be encoded in a separate "channel," so that in interconnecting several boxes of this kind each variable may be routed independently of the others. Figure 2.2 illustrates the case of an extension that is not componentwise. This box too "computes" a combinatorial function, but it is hard to see how the components of the input n -tuple could be made to come from different boxes, and those of the output n -tuple go to different boxes, without using complex encoders and decoders for which the problem of physical realizability would arise afresh.

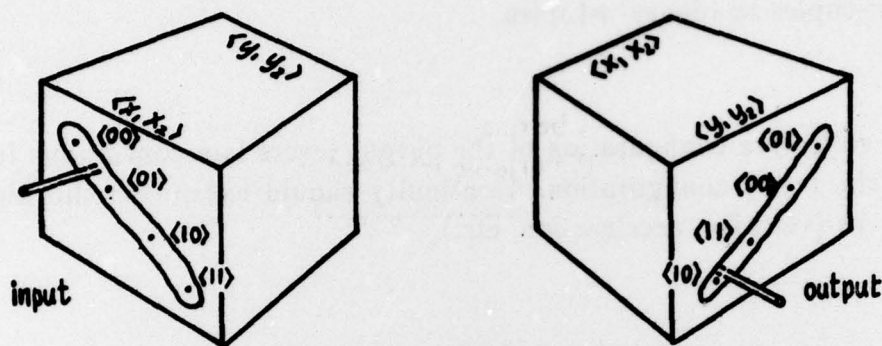


FIG. 2.2 An extension which is not componentwise. Only one degree of freedom is used to represent several binary variables.

3. Notation and mathematical preliminaries

We shall be dealing exclusively with functions that are *invertible*, and whose domain and range are *structured sets*, i.e., are explicitly given as indexed Cartesian products of sets. In particular, in all cases domain and range will be products of identical sets and will coincide.

A restriction of a function of the form $\Phi: \bar{A} \rightarrow \bar{B}$ is usually defined by specifying a subset A of the domain \bar{A} . However, when invertibility is an issue, it is necessary to explicitly specify also the restriction's intended range. Thus, by the *restriction of Φ to $\langle A, B \rangle$* (where $A \subseteq \bar{A}$ and $B \subseteq \bar{B}$) we shall mean the relation ϕ from A to B such that $a\phi b$ whenever $a \in A$, $b \in B$, and $\Phi(a) = b$. Whether ϕ is indeed a function, and an invertible one for that matter, depends on the choice of A and B . If ϕ is the restriction of Φ to $\langle A, B \rangle$, then Φ is an *extension of ϕ to $\langle \bar{A}, \bar{B} \rangle$* .

Given $\phi: A_1 \times \cdots \times A_m \rightarrow B_1 \times \cdots \times B_n$, an extension Φ of ϕ to $\langle P, Q \rangle$ is *componentwise* if there exist sets $\bar{A}_i \supseteq A_i$ and $\bar{B}_i \supseteq B_i$ such that $P = \bar{A}_1 \times \cdots \times \bar{A}_m$ and $Q = \bar{B}_1 \times \cdots \times \bar{B}_n$. In this case, ϕ is a *componentwise restriction of Φ* .

When the domain of a function is an indexed Cartesian product of sets, it is convenient to speak of *input variables* (or *input components*, or, simply, *arguments*) of the function, using the same indexing as for the corresponding sets. If also the *range* of the function is an indexed Cartesian product, one may likewise

speak of *output variables* (or *output components*) of the function. In ordinary function composition, an output variable of one function may be substituted for *any number* of input variables of other functions, i.e., "fan-out" is allowed, as indicated in Figure 3.1a. In what follows, we shall use a more restricted form of composition, called *one-to-one composition*, where any substitution of output variables for input variables must be one-to-one, as indicated in Figure 3.1b. If the output variable and the input variable involved in every such substitution range over identical sets, then one-to-one composition always yields invertible functions when applied to invertible functions.

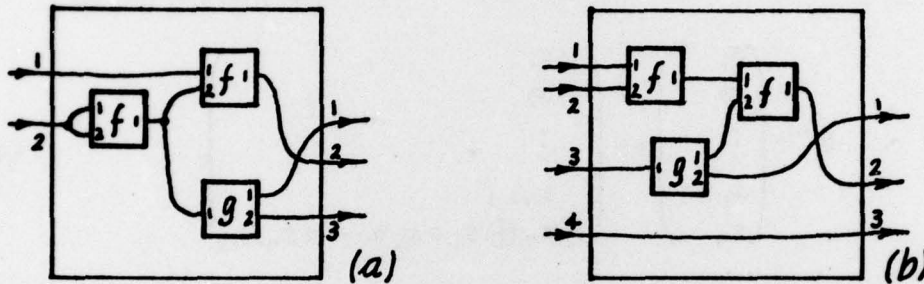


FIG. 3.1 (a) Examples of ordinary composition and (b) one-to-one composition of functions.

A re-indexing of input or output variables is a special case of one-to-one composition. One-to-one composition is conveniently handled by means of an algebraic notation formally analogous to that of tensor calculus[4]. From a physical viewpoint, the one-to-one constraint reflects the fact that signal fan-out requires a source of energy other than that carried by the signal itself.

Let ϕ be a binary relation from $S \times U_1 \times \cdots \times U_n$ to $S' \times U'_1 \cdots \times U'_n$, where the sets $U_1, \dots, U_n, U'_1, \dots, U'_n$ are *singletons*. For convenience, the one element of any of these singletons will be denoted by o . The variables associated with these singletons will be called *dummy*. A relation $\bar{\phi}$ from $S \times U_{i_1} \times \cdots \times U_{i_p}$ to $S' \times U'_{j_1} \times \cdots \times U'_{j_{p'}}$, where $1 \leq i_1 < \cdots < i_p \leq n$ and $1 \leq j_1 \leq \cdots \leq j_{p'} \leq n'$, is said to be obtained from ϕ by *deletion of dummy variables* if

$$\langle s, \overbrace{o, \dots, o}^n \rangle \phi \langle s', \overbrace{o, \dots, o}^{n'} \rangle \Leftrightarrow \langle s, \overbrace{o, \dots, o}^p \rangle \bar{\phi} \langle s', \overbrace{o, \dots, o}^{p'} \rangle,$$

that is, if the two relations coincide when the trailing o 's which accompany each tuple are disregarded.

Finally, a *combinatorial function* is one of the form $f: B^m \rightarrow B^n$, where B is the *binary set* $\{0, 1\}$.

4. Main results

DEFINITION 4.1 Consider the set $B = \{0, 1\}$ with the usual structure of Boolean ring, with " \oplus " (exclusive-or) denoting the addition operator, " \ominus " the additive-inverse operator (which in this case coincides with the identity operator), and " \circ " (AND) the multiplication operator. For any $n > 0$, the AND/NAND function of order n , denoted by $\theta^{(n)}: B^n \rightarrow B^n$, is defined by

$$\theta^{(n)}: \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ \ominus x_n \oplus x_1 \circ x_2 \circ \dots \circ x_{n-1} \end{pmatrix}. \quad (4.1)$$

REMARK 4.1 (a) The \ominus sign in (4.1), which is redundant (since $\ominus x_n = x_n$), has been introduced for symmetry with (4.2) below, where it is not redundant. (b) For any $n > 0$, $\theta^{(n)}$ is invertible and coincides with its inverse. (c) For $i = 1, 2, \dots, n-1$, the i -th component of $\theta^{(n)}$, i.e., $\theta_i^{(n)}$, coincides with the selector operator for the corresponding argument, i.e., $\theta_i^{(n)}(x_1, \dots, x_n) = x_i$. (d) The last component of $\theta^{(n)}$, i.e., $\theta_n^{(n)}$, coincides with the Boolean-complement operator for $n = 1$ (note that $x_1 \circ \dots \circ x_i = 1$ when $i = 0$), and with the exclusive-or of its two arguments for $n = 2$. (e) For all other values of n , $\theta_n^{(n)}$ is still linear in the n -th argument, but is *nonlinear* in the first $n-1$ arguments.

The family of AND/NAND functions was introduced by Toffoli[5] for proving the computation and construction universality of reversible cellular automata. An earlier, brief mention of the AND/NAND function of order 3 can be found in [2].

LEMMA 4.1. Any invertible combinatorial function of order n can be obtained by one-to-one composition of AND/NAND functions of order $\leq n$.

Proof. In the following construction we shall make use only of $\theta^{(n)}$ (where n is the order of the given function) and of $\theta^{(1)}$ (the Boolean-complement operator).

By definition, $\theta^{(n)}$ is a permutation on the set of n -tuples over B . (a) Any permutation can be written as the product of elementary permutations, i.e., of permutations that exchange only two n -tuples. In turn, as we shall prove below, (b) any elementary permutation on B^n can be written as the product of

atomic permutations, i.e., of permutations that exchange two n -tuples which differ in only one component. Observe that $\theta^{(n)}$ is the atomic permutation which exchanges $\langle 1, 1, \dots, 1, 0 \rangle$ with $\langle 1, 1, \dots, 1, 1 \rangle$. By reordering the components of $\theta^{(n)}$ and applying $\theta^{(1)}$ to selected components one obtains the family of all atomic permutations. Note that all the operations used above are forms of one-to-one composition. It remains to prove (b); this is done in the following way.

The n -tuples a_1, a_2, \dots, a_i are said to form a Gray-code path if two adjacent n -tuples differ by an atomic permutation. It is easy to verify that by means of sequence of atomic permutations the element at the beginning of the path can be moved to the end position, leaving the remainder of the path unchanged. By repeating such a move the first and last elements can be exchanged. The proof is completed by observing that any two n -tuples can be joined by a Gray-code path. \square

LEMMA 4.2 Consider the 1-manifold \hat{R} obtained by identifying all points of the real line R that differ by a multiple of 2π (\hat{R} can be thought of as the real circle), and let the points 0 and 1 of B coincide with, respectively, 0 and π of \hat{R} . Then there exists a diffeomorphism from \hat{R}^n to \hat{R}^n whose restriction to $\langle B^n, B^n \rangle$ coincides with $\theta^{(n)}$.

Proof. Consider \hat{R} with addition (" \oplus ") and additive inverse (" \ominus ") induced from those on R , and multiplication (" \circ ") defined as follows

$$x \circ y = \pi \frac{1 - \cos x}{2} \cdot \frac{1 - \cos y}{2}.$$

\hat{R} satisfies all the axioms for a ring except distributivity. Let $\Theta^{(n)}: \hat{R}^n \rightarrow \hat{R}^n$ be defined by

$$\Theta^{(n)}: \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ \ominus x_n \oplus x_1 \circ x_2 \circ \dots \circ x_{n-1} \end{pmatrix}. \quad (4.2)$$

Observe that when the operators defined on \hat{R} are restricted to $B \subseteq \hat{R}$ the Boolean-ring structure for B is recovered; thus, the restriction of $\Theta^{(n)}$ to $\langle B^n, B^n \rangle$ coincides with $\theta^{(n)}$. Moreover, $\Theta^{(n)}$ is infinitely differentiable by construction and coincides with its inverse; thus, $\Theta^{(n)}$ is a diffeomorphism. \blacksquare

As an immediate consequence of Lemmas 4.1 and 4.2, one obtains the following theorem (cf. GOAL 2.1).

THEOREM 4.1 Given any invertible combinatorial function $f^{(n)}: B^n \rightarrow B^n$, there exist a connected manifold $M \supseteq B$ and a diffeomorphism $F^{(n)}: M^n \rightarrow M^n$ such that $f^{(n)}$ is the restriction of $F^{(n)}$ to (B^n, B^n) .

5. Additional results

Before continuing with our mathematical exposition, it will be useful to verify in an intuitive way the physical realizability of the functions $\Theta^{(n)}$. With reference to Figure 2.1, we shall consider boxes whose input and output levers are constrained to circular motion (i.e., are cranks). In close correspondence with the defining formula (4.2), $\Theta^{(1)}$ will be realized as in Figure 5.1a, and $\Theta^{(2)}$ as in Figure 5.1b, where \oplus represent the mechanisms known as the "differential" which is used, for example, in automobile transmissions. In this mechanism, the angles p , q , and r satisfy the relation $q = -p + r$.

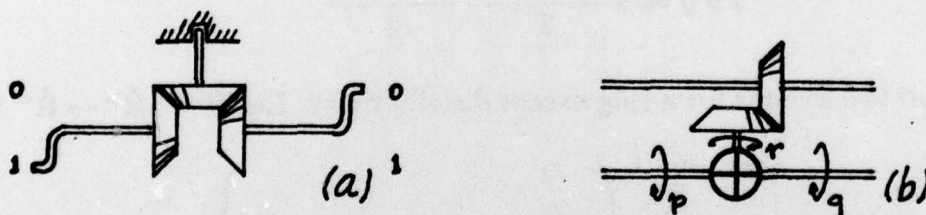


FIG. 5.1 (a) Realization of $\Theta^{(1)}$. (b) Realization of $\Theta^{(2)}$.

$\Theta^{(3)}$ will be realized as in Figure 5.2a, where the mechanism denoted by AND is illustrated in more detail in Figure 5.2b. Basically, the rotary motion of the two input shafts is converted to linear motion along two orthogonal axes x and y . The resulting composite motion operates a cam in whose two-dimensional surface the product of the two orthogonal displacements is encoded as a displacement along the z axis. A "finger" follows the surface of the cam and contributes an additive term to the differential.

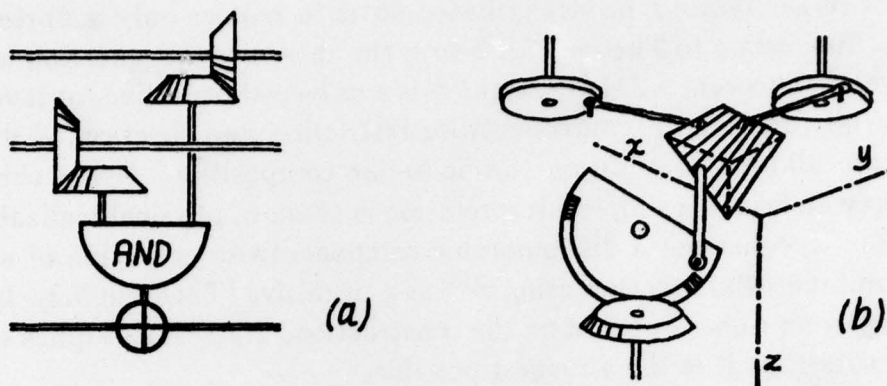


FIG. 5.2 (a) Realization of $\Theta^{(3)}$. (b) Details of the AND mechanism.

In general, $\Theta^{(n)}$ will be realized according to the scheme of Figure 5.3, which is convenient also for representing the corresponding discrete function $\theta^{(n)}$. The $(n - 1)$ -dimensional cam required for the $(n - 1)$ -input AND mechanism can be realized by cascading a suitable number of two-dimensional cams. Note that, although our construction makes use of rotary-to-linear conversion, which by itself is not an invertible operation and in general may introduce "dead points" in a mechanism, the resulting overall mechanism is indeed reversible.

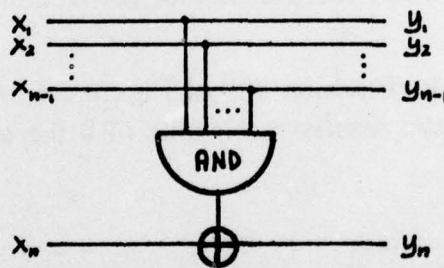


FIG. 5.3 Schematic representation of $\Theta^{(n)}$ or $\theta^{(n)}$

Returning to our mathematical exposition, let us observe that Lemma 4.1 supplies a set of invertible primitives for constructing—via one-to-one composition—any invertible combinatorial function. However, this set is unbounded, in the sense that θ 's of ever larger order may be needed as the order of the given invertible function increases. It is well known that any combinatorial function can be synthesized by ordinary function composition starting from a single computing primitive such as the two-input NAND function. In analogy

with this, can Lemma 4.1 be strengthened so as to require only a finite set of primitives? According to Theorem 5.1 below, the answer to this question is negative. However, Theorem 5.2 shows that $\theta^{(3)}$ is a universal primitive for invertible combinatorial functions if componentwise restriction and deletion of dummy variables are allowed in addition to one-to-one composition. Using the same operations (which have a simple interpretation in terms of physical realizability), it is possible to construct a diffeomorphic componentwise extension of any invertible combinatorial function using $\theta^{(3)}$ as a primitive (Theorem 5.3). In view of the many constraints imposed on the construction, this result is quite strong. We conjecture that it is the strongest possible.

THEOREM 5.1 *There exist invertible combinatorial functions of order n which cannot be obtained by one-to-one composition from AND/NAND functions of order $< n$.*

Proof. In the same context as the proof of Lemma 4.1, $\theta^{(i)}$ permutes exactly 2^{n-i} n -tuples of B^n . Thus, only even permutations can be obtained when $i > 0$. Since the product of even permutations is even, only even permutations can be obtained by one-to-one composition of any number of AND/NAND functions of order $< n$. \square

THEOREM 5.2 *Any invertible combinatorial function can be obtained by one-to-one composition, componentwise restriction, and deletion of dummy variables from $\theta^{(3)}$.*

Proof. Consider the function $\phi^{(5)}$ of Figure 5.4. A value of 0 for the fifth input component always results in a value of 0 for the corresponding output component.

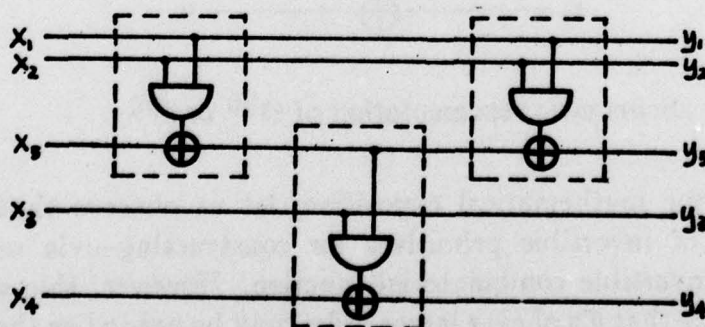


FIG. 5.4 Construction of $\phi^{(5)}$. When $x_5 = 0$, also $y_5 = 0$. The remaining components behave as the corresponding ones of $\theta^{(4)}$.

From the restriction of $\phi^{(5)}$ to $\langle B^3 \times \{0\}, B^3 \times \{0\} \rangle$ one obtains $\theta^{(4)}$ by deletion of the dummy variables x_5 and y_5 . In a similar way, all $\theta^{(n)}$ ($n > 3$) can be obtained. $\theta^{(2)}$ and $\theta^{(1)}$ are obtained directly from $\theta^{(3)}$ when the first and, respectively, the first two components are restricted to the value 1 and the resulting dummy variables are deleted. If one-to-one composition is applied before deletion, it is easy to verify that the number of deletions (i.e., the number of constant inputs) required for the construction of any invertible combinatorial function of order n does not exceed $2n - 3$. \square

THEOREM 5.3 For any invertible combinatorial function $f^{(n)}$, a diffeomorphic componentwise extension $F^{(n)}$ can be obtained by one-to-one composition, componentwise restriction, and deletion of dummy variables from $\Theta^{(3)}$.

Proof. The proof parallels that of Theorem 5.2. \square

6. Conclusions

Computing is based on the evaluation of functions that are *discrete* and *many-to-one*. On the other hand, the mechanisms offered by a schematization of physics such as classical mechanics are based on functions that are *continuous* and *one-to-one*. We have explicitly bridged the gap between these two conceptions.

Appendix

The question of whether there exist *reversible* systems (i.e., systems characterized by an invertible transition function) which possess universal computing capabilities has been considered by many authors (see [5] for references). The answer to this question is positive. For our purposes, it will be sufficient to recall the following basic proposition[3]:

For every combinatorial function $\phi: B^m \rightarrow B^n$ there exists an invertible combinatorial function $f^{(m+r)}: B^{m+r} \rightarrow B^{m+r}$ (with $r \leq n$) such that

$$\bigwedge_{1 \leq i \leq n} f_i^{(m+r)}(x_1, \dots, x_m, \overbrace{0, \dots, 0}^r) = \phi_i(x_1, \dots, x_m).$$

Informally, the required function ϕ is obtained from $f^{(m+r)}$ by assigning constant values to the r additional input components and ignoring the "random" values

obtained for the $m + r - n$ additional output components. (We use the term "random" for output values that depend on the first m input arguments and thus cannot be used as *constants* for a new computation. By contrast, the additional output components used in the proof of Theorem 5.2 yield "nonrandom" values.)

We cannot avoid mentioning the analogy of the above scheme of computation with the functioning of ordinary physical computers, where one must supply *work* (i.e., a nonrandom form of energy) in addition to the input signals, and remove *heat* (i.e., energy in random form) in addition to the output signals.

Acknowledgments

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